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Ralph E. Bach, Jr.,* and Yoram Baram**

SUMMARY

The approximate inversion of an internally unknown linear system, given by its impulse response sequence, by an inverse system having a finite impulse response, is considered. The recursive least squares procedure is shown to have an exact initialization, based on the triangular Toeplitz structure of the matrix involved. The proposed approach also suggests solutions to the problems of system identification and compensation.

1. INTRODUCTION

Linear system inversion--that is, interchanging the roles of the input and output signals--has been of interest in systems theory and in signal analysis in the past (see, e.g., [1], [2]). Recently, interest in the system inversion problem has been motivated by problems of geophysical exploration (see, e.g., [3]). In many cases, the system at hand is internally unknown and may not even possess a finite-order representation. An external representation in the form of the impulse response function may be generated, in principle, by measuring the response to a high-energy pulse, as is done in geophysical exploration, or by calculating the cross-correlation function between the output and a pseudo-random input, as is done in control applications. While, as we show, the impulse response of the inverse system is well defined, its calculation becomes practically intractable for long data records. Instead, a finite-order approximation of the inverse system may be sought. Such an approximation will yield, simply by inversion, a solution to the problem of system identification. We show that when the approximate inverse system is restricted to have a finite impulse response, the design problem can be solved by a recursive least-squares procedure having an exact initialization. The approach is readily extendable to the problem of cascade compensation.

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2. EXACT INVERSION

In order to demonstrate the need for approximation in the inversion of internally unknown linear systems, we first consider their exact inversion. Since discrete time systems often have delayed response, it is also our purpose to define a unified representation for the inverses of systems with and without delays that, without burdening the notation, will make the results applicable to both cases.

Let $\{h_k\}$, $k \geq 0$ be the impulse response of a causal time-invariant linear system. We shall assume for convenience that the system is single-input, single-output, and real, i.e., that $\{h_k\}$ is a scalar sequence. The system's output $\{y_k\}$ is obtained from its input $\{u_k\}$ by the convolution operation

$$y_k = h_k * u_k = \sum_{i=0}^k h_i u_{k-i} \quad k \geq 0 \quad (2.1)$$

It is readily verifiable that the inverse of (2.1) is given by

$$u_k = h_k^- * y_k \quad k \geq 0 \quad (2.2)$$

where $\{h_k^-\}$, $k \geq 0$ satisfies the progressive equation

$$h_k^- = -\frac{1}{h_0} \sum_{i=1}^k h_{k-i}^- h_i \quad k \geq 1$$

(2.3)

with

$$h_0^- = \frac{1}{h_0}$$

Next, suppose that $h_k = 0$ for $k = 0, \dots, l-1$, i.e., that the system has l time-unit delays. Then, clearly, the above inversion is ill-defined. The inverse system can, however, be redefined. First it should be realized that the inverse system no longer operates in "real time" on $\{y_k\}$ but, instead, it operates on a record of $\{y_k\}$, advanced l time-units with respect to the original sequence. Defining the sequences

$$y_k^{(1)} = y_{k+1}$$

and

(2.4)

$$h_k^{(1)} = h_{k+1} \quad k \geq 0$$

The system can, again, be written in "causal form" as

$$y_k^{(1)} = h_k^{(1)} * u_k \quad k \geq 0$$

which has the inverse

$$u_k = h_k^{(1)-} * y_k^{(1)} \quad k \geq 0 \quad (2.5)$$

where $(h_k^{(1)-})$, $k \geq 0$ satisfies

$$h_k^{(1)-} = -\frac{1}{h_0^{(1)}} \sum_{i=1}^k h_{k-i}^{(1)-} h_i^{(1)}, \quad k \geq 1$$

with

(2.6)

$$h_0^{(1)} = \frac{1}{h_1}$$

Since the 1-delay system now has exactly the same form as the 0-delay system, the index (1) can be removed with the understanding that when the system at hand has delays, its inverse must be interpreted as operating on $y_k^{(1)}$ as an input, as in equation (2.5).

It can be seen that the inverse of a causal, time-invariant, linear system given by an infinite-length impulse response is well defined. However, the calculation of the inverse impulse response becomes intractable, as both the memory and the amount of computations required grow indefinitely. Under certain conditions, discussed in the following section, the error resulting from taking a finite portion of the sequence (h_k^-) can be shown to diminish as the length of this portion is increased. However, this will not be, in general, an optimal choice of a finite impulse response approximation of the inverse system. Such approximation is the subject of the next section.

3. RECURSIVE LEAST-SQUARES INVERSION

Let $\{f_k\}$, $k \geq 0$, be the impulse response of an approximate model for $\{h_k\}$. Noting that

$$h_k^{-1} * h_k = \delta_k$$

where δ_k is the unit impulse sequence ($\delta_k = 1$ for $k = 0$, $\delta_k = 0$ for $k > 0$) an approximation of h_k^{-1} can be obtained by minimizing some measure of the difference between $f_k * h_k$ and δ_k with respect to f_k .

Denoting the sum-square norm of a sequence $\{a_k\}$, $k \geq 0$, by

$$||a_k|| = \left(\sum_{k=0}^{\infty} a_k^2 \right)^{1/2}$$

an approximation criterion for the problem at hand is defined by

$$\min_{\{f_k\}} ||f_k * h_k - \delta_k|| \quad (3.1)$$

Suppose that the inverse approximating model is restricted to have a finite impulse response of length n , that is, $f_k = 0$ for $k \geq n$. The approximation norm then becomes

$$||f_k * h_k - \delta_k|| = \left\{ \sum_{k=0}^{\infty} (h_k^T \underline{f} - b_k)^2 \right\}^{1/2} \quad (3.2)$$

where

$$\underline{f} = (f_0 \ f_1 \ \dots \ f_{n-1})^T$$

$$b_k = \begin{cases} 1 & k = 0 \\ 0 & k > 0 \end{cases}$$

and the n -dimensional vector sequence $\{h_k\}$ is given by

$$\underline{h}_k = \begin{cases} [h_k & h_{k-1} & \dots & h_0 & 0 & \dots & 0]^T & k \leq n-1 \\ [h_k & h_{k-1} & \dots & h_{k-n+1}]^T & k \geq n \end{cases} \quad (3.3)$$

The least-squares problem

$$\min_{\underline{f}} \sum_{k=0}^{\infty} (\underline{h}_k^T \underline{f} - b_k)^2$$

yields the equation

$$\underline{T}_\infty^T \underline{T}_\infty \underline{f} = \underline{T}_\infty^T \underline{b} \quad (3.4)$$

where

$$\underline{T}_\infty = \begin{bmatrix} \underline{h}_0^T \\ \underline{h}_1^T \\ \vdots \end{bmatrix} \quad \text{and} \quad \underline{b}_\infty = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \end{bmatrix}$$

Let $\underline{f}_m$ be defined by the equation

$$\underline{T}_m^T \underline{T}_{m-m} \underline{f}_m = \underline{T}_m^T \underline{b}_m \quad (3.5)$$

where, for $m \geq n$,

$$\underline{T}_m = \begin{bmatrix} \underline{h}_0^T \\ \underline{h}_1^T \\ \vdots \\ \underline{h}_{m-1}^T \end{bmatrix} = \begin{bmatrix} h_0 & & & 0 \\ h_1 & h_0 & & \\ \vdots & \vdots & \ddots & \\ h_{n-1} & h_{n-2} & \dots & h_0 \\ h_n & h_{n-1} & \dots & h_1 \\ \vdots & \vdots & & \\ h_{m-1} & h_{m-2} & \dots & h_{m-n} \end{bmatrix}$$

and

$$\underline{b}_m = [1 \ 0 \ \dots \ 0]^T$$

is m -dimensional. The matrix T_n can be seen to be lower triangular Toeplitz. It is readily verifiable (also see, e.g., [4]) that such structure yields the inverse

$$T_n^{-1} = \begin{bmatrix} h_0^- & & & 0 \\ h_1^- & h_0^- & & \\ \vdots & \vdots & \ddots & \vdots \\ h_{n-1}^- & h_{n-2}^- & \dots & h_0^- \end{bmatrix} \quad (3.6)$$

where, as in (2.3)

$$h_k^- = -\frac{1}{h_0} \sum_{i=1}^k h_{k-i}^- h_i \quad k \geq 1$$

and

$$h_0^- = \frac{1}{h_0}$$

Since h_0 can be assumed to be non-zero (otherwise, it is replaced by $h_0^{(t)} = h_t$, as explained in section 2), the matrix T_n^{-1} exists, yielding

$$\underline{f}_n = T_n^{-1} \underline{b}_n \quad (3.7)$$

For $m \geq n$, we obtain from the least-squares recursion (see, e.g., [5], p. 176)

$$\underline{f}_{m+1} = \underline{f}_m - K_m h_m^T \underline{f}_m \quad (3.8)$$

with

$$K_m = \frac{R_m h_m}{1 + h_m^T R_m h_m} \quad (3.9)$$

where $R_m = (T_m^T T_m)^{-1}$ satisfies the recursion

$$R_{m+1} = R_m - R_m \frac{h_m h_m^T}{1 + h_m^T R_m h_m} R_m \quad (3.10)$$

initialized at

$$R_n = T_n^{-1} (T_n^{-1})^T \quad (3.11)$$

where T_n^{-1} is given by (3.6). We note that while it is often suggested that a recursive least-squares procedure be initialized at arbitrary values of R_0 and $\underline{f}_0$ in order to avoid matrix inversion (see, e.g., [5], p. 177), the above procedure is initialized at the exact values of $\underline{f}_n$ and R_n , specified by (3.7) and (3.11).

The convergence of the above recursive procedure is examined next. From (3.4) we directly obtain the equation

$$R \underline{f} = \underline{h}_0$$

where $\underline{h}_0$ is defined by (3.3) and

$$R = T_m^T T_m = \begin{bmatrix} a_0 & a_1 & a_2 & \cdots & a_{n-1} \\ a_1 & a_0 & a_1 & \cdots & a_{n-2} \\ a_2 & a_1 & a_0 & \cdots & a_{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n-1} & a_{n-2} & a_{n-3} & \cdots & a_0 \end{bmatrix}$$

where

$$a_i = \sum_{k=0}^{\infty} h_k h_{k+i}$$

Suppose that the system at hand is externally (bounded input/bounded output) stable. Then

$$\sum_{k=0}^{\infty} |h_k| < \infty$$

we have

$$\sum_{k=0}^{\infty} |h_k h_{k+1}| = \sum_{k=0}^{\infty} |h_k| |h_{k+1}| \leq \sum_{k=0}^{\infty} |h_k| \sum_{k=0}^{\infty} |h_{k+1}| < \infty$$

yielding convergence of a_i . It follows that the recursive least-squares inversion procedure (3.8) converges if the system is externally stable.

The inversion error at each step of the recursive inversion procedure can be calculated as

$$\epsilon_m^2 = (T_m f_m - b_m)^T (T_m f_m - b_m)$$

which, by substituting (3.5), yields

$$\epsilon_m^2 = 1 - f_{m,0} h_0$$

where $f_{m,0}$ is the first component of f_m .

We next examine the behavior of the inversion error as the length (or the order) of the inverse system is increased to infinity. We assume that the sequence (h_k^-) is bounded (in other words, the inverse system is stable). Let us define the sequence

$$p_k = \begin{cases} h_k^- & k = 0, \dots, n-1 \\ 0 & k \geq n \end{cases}$$

Then, since $\{f_k\}$ minimizes the inversion norm, we have

$$||f_k * h_k - \delta_k|| \leq ||p_k * h_k - \delta_k||$$

yielding

$$\lim_{n \rightarrow \infty} ||f_k * h_k - \delta_k|| \leq ||h_k^- * h_k - \delta_k|| = 0$$

hence,

$$\lim_{n \rightarrow \infty} \|f_k * h_k - \delta_k\| = 0$$

In words, the inversion error diminishes to zero as the length of the inverse filter is increased to infinity.

An approximate model for the system at hand can be obtained by inverting the resulting inverse system. The model will be of the all-pole type, specifically $1/(f_0 + f_1 z^{-1} + \dots + f_{n-1} z^{-(n-1)})$, where z is the z -transform variable. We note that if the given system has, say, l time unit delays, that is, $h_i = 0$ for $i = 0, \dots, l-1$, the resulting approximate model should be multiplied by z^{-l} , as the proposed technique, in effect, approximates the given system multiplied by z^l . This approach yields a solution to the problem of system identification. The impulse response sequence may be obtained directly by applying a high-energy pulse at the input, or, by simple calculation, from the cross correlation between the output and a white noise input.

Next, suppose that it is desired to compensate a system given by its impulse response sequence $\{h_k\}$ by cascading it with a relatively simple system so as to approximate a desired system whose impulse response $\{g_k\}$ is given or can be calculated. When the compensator is restricted to have a finite impulse response, its parameters may be obtained by minimizing the norm $\|f_k * h_k - g_k\|$. The recursive least-squares procedure (3.8) is then replaced by $\hat{f}_{m+1} = \hat{f}_m + K_m (g_m - h_m^T \hat{f}_m)$, with K_m and $\hat{f}_m$ defined by (3.9-3.11) and (3.3), respectively. Finally, we note that a recent paper [6] addresses the least-squares compensation, inversion, and approximation of internally known linear systems by polynomial systems.

4. CONCLUSION

The inversion of an internally unknown linear system has been considered. A recursive least-squares procedure having an exact initialization, for calculating the parameters of a finite impulse response inverse system, has been derived. The proposed approach also suggests solutions to the problems of system identification and compensation.

REFERENCES

1. L. A. Zadeh and C. A. Desoer, Linear System Theory, McGraw-Hill, New York, 1963.
2. A. V. Oppenheim and R. W. Schaffer, Digital Signal Processing, Prentice-Hall, Englewood Cliffs, NJ, 1975.
3. E. A. Robinson and S. Treitel, "Geophysical Signal Analysis," Prentice-Hall, Englewood Cliffs, NY, 1980.
4. D. Commenges and M. Monsion, "Fast Inversion of Triangular Toeplitz Matrices," IEEE Trans. on Automatic Control, Vol. AC-29, No. 3, March 1984.
5. G. C. Goodwin and R. L. Payne, Dynamic System Identification, Academic Press, New York, 1977.
6. Y. Baram, "Polynomial Compensation, Inversion and Approximation of Discrete Time Linear Systems," NASA TM 100027, 1987.



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